

Rate of parity violation from measure concentration

NIKOS KALOGEROPOULOS

Department of Science
BMCC-The City University of New York
199 Chambers St., New York, NY 10007, USA

Abstract

We present a geometric argument determining the kinematic (phase-space) factor contributing to the relative rate at which degrees of freedom of one chirality come to dominate over degrees of freedom of opposite chirality, in models with parity violation. We rely on the measure concentration of a subset of a Euclidean cube which is controlled by an isoperimetric inequality. We provide an interpretation of this result in terms of ideas of Statistical Mechanics.

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E-mail: nkalogeropoulos@bmcc.cuny.edu
nkaloger@yahoo.com

1. INTRODUCTION

The rate at which a system with many degrees of freedom approaches its equilibrium state [1],[2] is one of its most important characteristics, ranking alongside the existence and the determination of the equilibrium configuration itself. Arguments determining such rates have appeared since the earliest days of statistical mechanics [1] and have also played an important role in classical and quantum field theories [3],[4]. There is a far greater variety of non-equilibrium than equilibrium phenomena, in which even conventional thermodynamic quantities and relations may need redefinition [5]-[8]. For such a reason alone, it may be of some importance to be able to determine rates of convergence to equilibrium in an, as much as possible, model-independent way. In this paper we attempt to determine such a rate of convergence for a system exhibiting parity violation.

Parity violation is one of the most intriguing, experimentally verified effects encountered in physics. This phenomenon has been a subject of extensive studies due to its great importance for particle physics [9]-[13], nuclear physics [14]-[16], quantum gravity [17]-[19], astrophysics [20] and atomic physics [21].

In many instances the detailed mechanism of parity violation is not understood [9]-[13] at a fundamental level. There are other cases, effective field theories being an example, where it may not even be important to know the fundamental underpinnings of such a behavior. In either case, conclusions relying on general kinematic grounds, which are largely independent of the specific interactions of the model at hand, may be of interest in determining general characteristics of the behavior of the degrees of freedom under consideration.

An outline of this paper is as follows: In Section 2, we construct a simple geometric model that quantifies phase-space contribution to the rate with which the asymmetric production of the two chirality degrees of freedom leads to a dominance of the one kind over the other. We find that this factor of the rate of dominance has a Gaussian behavior, when the number of degrees of freedom n is very large. In Section 3, we provide a physical explanation for this emergent Gaussian behavior by associating it to the rate at which a classical statistical system approaches its equilibrium state. In Section 4, we draw some conclusions, provide the context and make some additional comments regarding the applicability of this approach.

2. A GEOMETRIC MODEL AND GAUSSIAN CONCENTRATION

We assume from the outset the existence of a mechanism of parity violation without attempting to provide any details of its origin or its specific workings.

Consider a field ψ with a definite chirality. Then ψ is an eigenfunction of the chirality operator with eigenvalues ± 1 . In particle physics [9], for instance, the chirality of space-time fermionic degrees of freedom is determined by the eigenvalue of ψ under the action of $\frac{1}{2}(1 \pm \gamma_5)$, as

$$\frac{1 \pm \gamma_5}{2} \psi = \pm \psi \quad (1)$$

where γ_5 denotes the top element of the Clifford algebra of gamma matrices [9]. We indicate the chirality eigen-states of ψ by $|+\rangle$ and $|-\rangle$ in analogy with the spin- $\frac{1}{2}$ notation. Let the field theoretical model have n such degrees of freedom. Eventually n should be considered very large, namely, we will be interested in the “thermodynamic” limit $n \rightarrow \infty$. The possible chirality eigen-states form a basis of a tensor product representation of $\mathbb{Z}_2^{\otimes n}$, where $\mathbb{Z}_2 = \{-1, +1\}$ and as such they have the form

$$|p_1\rangle \otimes |p_2\rangle \otimes \cdots \otimes |p_n\rangle \quad (2)$$

where p_i , $i = 1, 2, \dots, n$ stand for $+$ or $-$ in this equation and for the rest of this paper. The carrier space of such a representation is the Hilbert space of the n -particle states \mathcal{H}_n . Let us focus exclusively on the chirality behavior of the degrees of freedom ψ and forget any other properties that they may possess. Then \mathcal{H}_n has a basis comprised of elements of the form (2). Geometrically, \mathcal{H}_n can be seen as the set of vertices of the Euclidean cube \mathbb{E}_2^n in \mathbb{R}^n whose side has a length of 2 units. This cube is the phase space pertinent to the parity degrees of freedom of the system.

Consider now two n -particle chirality eigen-states $|P\rangle = |p_1\rangle \otimes |p_2\rangle \otimes \cdots \otimes |p_n\rangle$ and $|P'\rangle = |p'_1\rangle \otimes |p'_2\rangle \otimes \cdots \otimes |p'_n\rangle$. These two eigen-states can be geometrically represented as two vertices of the cube \mathbb{E}_2^n . We would like to find a reasonable expression for a “distance” between such states/vertices. This is equivalent to choosing a metric for \mathbb{E}_2^n . One reasonable choice would be the induced metric on \mathbb{E}_2^n from the Euclidean metric of \mathbb{R}^n . A potential drawback of such a choice, which is induced by the l^2 norm, is that it does not only express intrinsic properties of \mathbb{E}_2^n but also reflects, to some extent, the

metric embedding $\mathbb{E}_2^n \hookrightarrow \mathbb{R}^n$. For this reason, as well as for greater simplicity, we prefer to avoid such a choice, and use instead the normalized Hamming metric $d_n : \mathbb{E}_2^n \times \mathbb{E}_2^n \rightarrow \mathbb{R}_+$ instead, which is defined by

$$d_n(|P\rangle, |P'\rangle) = \frac{1}{2n} \sum_{i=1}^n |p_i - p'_i| \quad (3)$$

where $|\cdot|$ indicates the absolute value in \mathbb{R} . This choice corresponds to the choice of an l^1 norm for \mathbb{E}_2^n up to a normalizing constant $1/2n$. It is easy to check that d_n satisfies all three properties (positivity, symmetry and the triangular inequality) required of metrics. We can define, following the spirit of (3), the distance between a vertex $|x\rangle \in \mathbb{E}_2^n$ and the set $A \subseteq \mathbb{E}_2^n$ as

$$d_n(|x\rangle, A) = \inf\{d_n(|x\rangle, |P\rangle), \forall |P\rangle \in A\} \quad (4)$$

Continuing in this mode, we define the ϵ -extension of A [22], [23] as

$$A_\epsilon = \{|x\rangle \in \mathbb{E}_2^n : d_n(|x\rangle, A) \leq \epsilon, \epsilon > 0\} \quad (5)$$

Then A_ϵ is the set of all points (vertices) of \mathbb{E}_2^n which are at a Hamming distance at most ϵ away from A . Since \mathbb{E}_2^n is a discrete set, it is reasonable to consider as the measure (the “volume”) of $A \subseteq \mathbb{E}_2^n$ its cardinality $\text{card}(A)$. We will eventually be interested in a comparison of the number of states (vertices) of the phase space \mathbb{E}_2^n , representing the dominant chirality, with the total number of states (vertices) of \mathbb{E}_2^n . In light of this future comparison, it is more advantageous to work with the normalized cardinality $P(A)$ of A instead, defined as

$$P(A) = \frac{\text{card}(A)}{2^n} \quad (6)$$

For concreteness, we can assume, without any loss of generality, that A is a set of vertices of \mathbb{E}_2^n which represent states with more (or equal to) positive than negative chirality degrees of freedom. Obviously $P(A) \geq \frac{1}{2}$. For such A , [22], [23] defines the concentration function

$$h(A, \epsilon) = 1 - \inf\{P(A_\epsilon), A \subseteq \mathbb{E}_2^n, \epsilon > 0\} \quad (7)$$

which measures the maximal fraction of the “volume” of \mathbb{E}_2^n not belonging to A_ϵ . In our particular case, ϵ is proportional to the difference of the number of states in phase

space of the dominant minus the recessive chirality. It was proved in [24], [25] by relying on an isoperimetric inequality on graphs, that

$$h(A, \epsilon) \leq \frac{1}{2} e^{-2\epsilon^2 n} \quad (8)$$

This result can be interpreted as stating that, as the number of degrees of freedom n increases without any upper bound, more states of the phase space \mathbb{E}_2^n are concentrating closer and closer to A_ϵ . This rate of concentration is given approximately by $\epsilon \sim n^{-\frac{1}{2}}$. This argument also implies that A_ϵ will eventually contain almost all the vertices of \mathbb{E}_2^n as $n \rightarrow \infty$. Therefore, chirality eigen-states containing more positive than negative chirality degrees of freedom dominate so fast, that the recessive chirality eigen-states disappear exponentially fast with n . The concentration function (7) contributes an exponential factor to the rate of dominance of one chirality over the other. Another factor to such a rate will be contributed by the specific dynamics of the model at hand. Such dynamics will be manifested through the paths, expressing the time evolution, connecting the vertices that the system will occupy in the phase space \mathbb{E}_2^n . We also observe in (8) that the concentration function is Gaussian with respect to ϵ . Gaussian functions are encountered very frequently as a result of the law of large numbers: the present result is no exception. What we have just found is a geometric incarnation, in our specific context of the law of large numbers [26].

It is probably worth mentioning at this point, that, using the theory of large deviations, one can arrive, through a different line of reasoning, to the following upper bound for the concentration function [22], [23]

$$h(A, \epsilon) \leq 2 e^{-\frac{\epsilon^2 n}{16}} \quad (9)$$

In neither of these cases is the upper bound claimed to be optimal. Roughly speaking, there is very little difference between (8) and (9). What really matters the most is the convergence rate of $h(A, \epsilon)$ in terms of $\epsilon^2 n$, regardless of its pre-factor in (8) or (9).

The convergence behavior of $h(A, \epsilon)$, as $n \rightarrow \infty$, is physically important because as does not depend on the value of the production rate or on the specifics of the interactions of the degrees of freedom of a given chirality. This is to be expected of course, as this factor is due only to kinematics (phase space) and ignores the specific dynamics of the model. As long as the degrees of freedom of one chirality are produced at a higher rate

than the other, with the production rate difference ϵ being constant in time, one chirality will asymptotically dominate over the other at a rate which has a Gaussian factor involving the excess ϵ . We also assume that ϵ should not approach zero faster than $n^{-\frac{1}{2}}$, if results (8) or (9) are to hold.

3. ON THE ORIGIN OF THE GAUSSIAN BEHAVIOR

The emergence of a Gaussian dependence of $h(A, \epsilon)$ in terms of ϵ may be unexpected, so in this Section we attempt to elucidate it. This procedure has two stages. Both of them are inspired and follow the spirit of the analysis of Boltzmann on the derivation of the thermodynamic properties of an ideal gas [27]. We also stress the underlying geometric structure as much as possible.

First, consider the unit sphere S^n . Its volume is given [28] by

$$\text{vol}(S^n) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} \quad (10)$$

Using the Stirling approximation [29], for $n \rightarrow \infty$

$$\Gamma(n) = \sqrt{\frac{2\pi}{n}} \left(\frac{n}{e}\right)^n \quad (11)$$

we can see that the asymptotic behavior of $\text{vol}(S^n)$ is given by

$$\text{vol}(S^n) \sim (n+1)^{-\frac{n+1}{2}} \quad (12)$$

The cube \mathbb{E}_2^{n+1} with the normalized Hamming distance (3) has diameter 1. To draw a clearer analogy with the case of S^n , we consider, instead, the comparison with a Euclidean cube \mathbb{E}^n of side 2 units. To be able to make a reasonable comparison with S^n , we re-scale the diameter (length of longest diagonal) of \mathbb{E}^{n+1} to 1 unit, which amounts to a re-scaling of its side length to $(n+1)^{-\frac{1}{2}}$. Then

$$\text{vol}(\mathbb{E}^{n+1}) = (n+1)^{-\frac{n+1}{2}} \quad (13)$$

We observe that the leading term in the asymptotic behavior of the volumes of S^n and \mathbb{E}^{n+1} is the same, as $n \rightarrow \infty$. Therefore in this approximation, we can use S^n instead of \mathbb{E}^{n+1} in arguments involving volumes, as is done in the next paragraphs.

The substitution of \mathbb{E}_2^n endowed with the Hamming metric by \mathbb{E}^n endowed with the Euclidean metric, which was performed in the preceding paragraph can also be justified more formally, without altering the essence of the argument above, as follows: We can look at the “dual description” of the two cubes we compare, in terms of the behavior of the set of real functions that are defined on them. Let $h : \mathbb{R}^n \rightarrow \mathbb{R}$. We note that these cubes are also the unit spheres with respect to the usual l^1 and l^2 norms. The corresponding norms on the spaces of Lebesgue-integrable functions L^1 and L^2 of these cubes are denoted by $\|h\|_1$ and $\|h\|_2$, respectively. Almost all functions encountered in particle physics are analytic, at least in part of their domain of definition, so the Taylor series expansion of such functions receives its dominant contribution from their linear term, as long as the function is not expanded around one of its critical points. For such linear approximations of h , a combination of the Hölder and the Kahane-Khinchine inequalities gives [30]

$$\|h\|_1 \leq \|h\|_2 \leq \sqrt{2} \|h\|_1$$

Therefore, the two cubes that we are considering with the Euclidean and the Hamming metrics are quasi-isometric. So, if we look at their metric properties, they are essentially the same, “very roughly speaking”. Since our explanation of the emergence of the Gaussian behavior is only very rough, in the sense that as $n \rightarrow \infty$, deviations of order $n^{\frac{1}{2}}$ or smaller are inconsequential, as seen from (8),(9), such a quasi-isometric equivalence is sufficient for our purposes.

As a second step we focus, as in the previous paragraph, on the dual description [22], [23]: we consider the set of all continuous, square-integrable functions $f_i : S^n \rightarrow \mathbb{R}$, $i = 1, \dots, n+1$. Most of the physical functions of interest are assumed to be smooth enough, and they satisfy these conditions. Actually, this type of argument has been developed for Lipschitz functions [26]. We should consider, once more, S^n as embedded in \mathbb{R}^{n+1} . Let $x_i \in \mathbb{R}$, $i = 1, 2, \dots, n+1$ be the Cartesian coordinates in \mathbb{R}^{n+1} and $r^2 = \sum_{i=1}^{n+1} x_i^2$ indicate the radial distance from the origin of this coordinate system. Let $dvol$ stand for the Riemannian volume element of S^n with respect to the round metric. By using the well-known facts [29]

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2}} dt = 1 \quad (14)$$

and

$$\prod_{i=1}^{n+1} dx_i = r^n dr dvol \quad (15)$$

we find

$$\int_0^\infty e^{-\frac{r^2}{2}} r^n dr = \frac{(2\pi)^{\frac{n+1}{2}}}{\text{vol}(S^n)} \quad (16)$$

We then get

$$\int_{S^n} \left\| \sum_{i=1}^{n+1} x_i f_i \right\|^2 d\text{vol} = \frac{\text{vol}(S^n)}{(2\pi)^{\frac{n+1}{2}}} \int_{S^n} \left\| \sum_{i=1}^{n+1} x_i f_i \right\|^2 d\text{vol} \int_0^\infty r^n e^{-\frac{r^2}{2}} dr \quad (17)$$

which can be rewritten as

$$\frac{1}{\text{vol}(S^n)} \int_{S^n} \left\| \sum_{i=1}^{n+1} x_i f_i \right\|^2 d\text{vol} = \frac{1}{(2\pi)^{\frac{n+1}{2}}} \int_{\mathbb{R}^{n+1}} \left\| \sum_{i=1}^{n+1} x_i f_i \right\|^2 e^{-\sum_{i=1}^{n+1} \frac{x_i^2}{2}} dx_1 \cdots dx_{n+1} \quad (18)$$

which eventually gives

$$\frac{1}{n+1} \langle \left\| \sum_{i=1}^{n+1} s_i f_i \right\|^2 \rangle = \frac{1}{\text{vol}(S^n)} \int_{S^n} \left\| \sum_{i=1}^{n+1} x_i f_i \right\|^2 d\text{vol} \quad (19)$$

where s_i , $i = 1, \dots, n+1$ are Gaussian random variables with mean 0 and variance $\sigma = 1$. Here $\langle \cdot \rangle$ denotes the Gaussian average of its argument over \mathbb{R}^{n+1} , hence the appearance of the $n+1$ in the denominator. The conclusion is that averages of continuous, square integrable functions $f_i, i = 1, \dots, n+1$ over S^n can be expressed as Gaussian averages of these functions over \mathbb{R}^{n+1} [22],[23],[26],[27].

The previous paragraph applies, to a great extent, for functions on S^n . However, if one is not interested in the very “small-scale” details of the geometry of S^n , then one may choose to “probe” S^n with objects providing a “resolution” of order at most ϵ . The smallest distances x that we want to be able to distinguish in this argument are of order of magnitude $n^{-\frac{1}{2}}$, which is the approximate length of the side of the cube \mathbb{E}^{n+1} , having the same diameter as S^n . Unambiguous results can be drawn as long as $\epsilon = \sigma \cdot \frac{1}{\sqrt{n}}$, which amounts to $\epsilon\sqrt{n} = \sigma$. Then S^n and \mathbb{E}^{n+1} do not only have the same volume (as $n \rightarrow \infty$), but also appear, roughly, metrically the same at a scale of $x \sim \sigma$ or larger. Within such a minimum length scale rough approximation, the previous analysis of the Gaussian behavior of functions on S^n applies equally well to \mathbb{E}^{n+1} . So

$$\frac{1}{n+1} \langle \left\| \sum_{i=1}^{n+1} s_i f_i \right\|^2 \rangle = \int_{\mathbb{E}^{n+1}} \left\| \sum_{i=1}^{n+1} x_i f_i \right\|^2 d\mu_{\mathbb{E}^{n+1}} \quad (20)$$

where $\| \cdot \|$ stands for the L^2 norm in the space of interest in (17)-(20) and s_i has the form

$$s_i \sim e^{-\epsilon^2 n} \quad (21)$$

which agrees, approximately, with the results of (8) and (9).

It is worth noticing, at this point, that there is nothing really stochastic about the chirality eigen-states. The corresponding chirality eigen-values are ± 1 with probability 1. This deterministic behavior can be formally recovered in the limit of shrinking the “resolution” $\epsilon \rightarrow 0$, which amounts to re-scaling the variance of the Gaussian variables s_i , so that it approaches zero, since

$$\delta(x) = \lim_{\sigma \rightarrow 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \quad (22)$$

4. DISCUSSION AND CONCLUSIONS

In this work we presented a geometric argument determining the phase-space contribution (kinematics) to rate of dominance of degrees of freedom of a given chirality over their recessive counterparts. We used the fact that parity is a \mathbb{Z}_2 symmetry to construct the model cube of the argument and used an isoperimetric inequality to determine that the rate of dominance has a Gaussian behavior for a large number of degrees of freedom. We observe that the same kinematic argument can be applied in determining the phase-space contribution to the rate of approaching equilibrium for any statistical system possessing a degree of freedom that can take two possible values. Although, for concreteness, we used the parity violation as an example in this paper, the argument can be repeated to any such two-state systems as well.

The spirit of the argument leading to (19) can be traced back to Boltzmann [27]. It was reached by him in an attempt to derive statistically the thermodynamic properties of classical ideal gases from their constituent properties. It was subsequently conjectured by him to cover all systems, as long as they obey the ergodic hypothesis [27]. Consequently, it lies at the heart of the equality of the averages obtained by using either the micro-canonical or the canonical ensemble in deriving the thermodynamic properties of equilibrium systems from their microscopic or mesoscopic constituents [27]. This argument is also extensively used in derivations involving noise in Langevin-type equations, especially when reexpressing the stochastic dynamics through the functional formalism [3],[4]. P. Lévy derived these results in a probabilistic setting [22],[23],[26]. Subsequently, they were used extensively in [22],[23] in a functional analytic and in [26], in a geometric context. The present work lies at the confluence of some of these ideas and also provides

a direct application of isoperimetric inequalities [31], which are used very extensively in geometry, in a more physically relevant context.

Although it may be possible to derive (8),(9) using better-known analytical techniques, we believe that the method that we follow here may enhance our intuition about the underlying reasons determining the rate of convergence of a system toward its equilibrium state. Moreover, due to the relative insensitivity of the isoperimetric inequalities [31] and the concentration of measure phenomenon [22],[23] to the degree of smoothness of the functions employed, as long as they are Lipschitz, the present approach may also be applicable in instances where the more conventional analytic methods may conceivably not be as effective.

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